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THE MOST PROBABLE, THE MOST POSSIBLE

By

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Abstract

The work is focused on the application of possibility theory to property appraisal. The appraiser is usually required to find the "most probable selling price" of the property to be estimated in an uncertain environment. In other sciences uncertain information has been analyzed in different ways. Starting from the possibility theory (L. Zadeh, 1978) it is possible to deal with property market data coming, in some contexts, to different conclusions.

The work is organized as follows: after an introduction, the first paragraph will analyze a brief profile of fuzzy sedts and theory of possibility. In the second paragraph the attention will be concerned about the comparison between the most probable selling price and the "most possible selling price". A final paragraph will offer final remarks and future directions of research.

Introduction[1]

Although the great development of the analysis regarding uncertain problems, the property appraisal relies on the probabilistic definition of the "most probable selling price". The main issue of this work is the application of a different approach to the uncertainty. There is a growing interest in a field of study called "Imprecise Probability".[2] This is a generic term to cover all the mathematical models which measure the chance or the uncertainty without any sharp numerical probabilities.

In particular, this work would analyze the contribution of the "possibility theory" based on fuzzy sets developed by Lotfi Zadeh[3] to appraisal problems. Fuzzy logic has been applied to the real estate evaluation process[4]. The definition of the "most possible selling price" based on this theory will be highlighted and compared to the most widespread definition of the "most probable selling price". The work is organized as follows: in the following first paragraph the attention will be focused on fuzzy logic, fuzzy sets and the theory of possibility. The second paragraph will concern an application of the concept of the most possible selling price to a real estate appraisal. Final remarks and future directions for the research will be given in the final paragraph.

1. Fuzzy Logic, Fuzzy Sets, and the Theory of Possibility : an overview

The Aristotelian or bivalent logic is essentially based on two principles: the **Law of Contraddiction**, which states that an element could not belong, at the same time, to a set and its complement. The second **Law of the Excluded Middle** states that the union between a set and its complement is equal to the universe to which they belong.

These laws are the basis of the formal logic. On this paradigm, most of computer programming and mathematical reasoning is based. Probability concept is based on these premises, too.

According to these premises a set of comparable properties could be written as follows:

$$A = \{x \mid x \text{ is a property comparable to } z\} \qquad (a)$$

The elements x could belong or not to the set, there are not any other alternatives . According to these laws it is possible to indicate a "membership rule" μ in order to define the membership or not to a classic set. For example, it defines the set of a real property comparable to a real property z. As a consequence the set A can be pointed out. The elements x could belong or not to the set, there are not any other alternatives.

Then the membership function will assume different values in the two different cases:

$$\mu_A(x) = 1 \text{ for } x \in A \quad \text{and} \quad \mu_A(x) = 0 \text{ for } x \notin A \qquad (b)$$

An element can only belong or not to A set if it has one (or more) specific characteristic. If the real property appraiser refers to a subjective probability he will define the probability of an occurrence of each price of any "comparable" properties. Different degrees of probability as 10% or 50% can be defined, but the comparable property can only belong or not to each degree of probability.

An element can only belong or not to A set, a property has or does not have a certain percentage of comparability to the property to be estimated. In 1965 Lotfi Zadeh[5] defined for the first time, a different kind of sets: fuzzy sets. The relationship between the set and the element, was defined upon a membership function that allows also different "*grades of membership*" not only the Boolean variables 1 and 0 In his point of view, these different degrees of membership are given in several ways. Five methodologies can be taken into account[6]. The first is based on "**subjective evaluation and elicitation".** It can be considered the most frequent case and it is essentially based on the appraiser's cognitive states defined through simple or sophisticated elicitation procedures. In this case they "*would be (usually) arrived at almost instantaneously without any conscious analysis*"[7]. Another way is the "**ad hoc forms**". In this case the decision starts from a predefined slope of a fuzzy number. The appraiser has to choose only the central values and the slope on both sides. Sometimes it is possible also to **convert probability frequency histogram into possibility degrees** of membership. It must be highlighted that there are several different methods able to transform probability histogram into possibility measures. This is possible even if the concept of probability is very different from the possibility one. It is possible to calculate the membership degrees through a "**physical measurement"**, too. In this way the physical measures of a particular phenomenon allow to define a degree of membership. The last method to define a degree of membership can be considered **"learning and adaptation"**. In this case the membership degrees are adapted to the reality by a learning process.

The concept of membership is not longer crisp (either 1 or 0), but can have several "*degrees of membership".*

It is possible to rewrite the (a) formula as:

$$A = \{ (x, \mu_A(x)) \mid x \in A, \mu_A(x) \in [0,1] \} \qquad (c)$$

For example a set composed by three comparable real property prices can be defined through the following fuzzy set:

$$\boldsymbol{A} = \{ (a, 0.1), (b, 0.5) (c, 0.9) \} \qquad (d)$$

These degrees of membership are included between 0 (there is not any probability of currency of that price) and 1 (it is the price that will occur with the highest degree of "possibility"). In spite of a classic set they allow us to deal with data having "different grades of comparability".

Reasoning with this "many value" logic could improve the significance of the number in an appraisal report, making them closer to the reality. They are the so-called fuzzy numbers. There has been an application to the reconciliation phase of a property appraisal[8].

Starting from the concept of a fuzzy set a "**possibility distribution**" can be defined. In his fundamental work L. A. Zadeh wrote "…*A thesis advanced in this paper is that when our main concern is with the meaning of information – rather than with its measure, the proper framework for information analysis is possibilistic rather that probabilistic in nature, thus implying that what is needed for such an analysis is not probability theory but an analogous – and yet different- theory which might be called the* ***theory of possibility****…*"[9]

The term "possibilistic" as quoted in that work was defined for the first time in a famous work of Gaintes and Kohout[10].

This work would highlight that in real estate market "…*much of the information on which human decision are based, is more* ***possibilistic*** *than probabilistic …the intrinsic fuzziness of natural languages - which is logical consequence of the necessity to express information in a summarized form - is, in the main,* ***possibilistic*** *in origin…*" [11]

The consumer behaviour and the property prices are affected by the "fuzziness of natural language". A possibility distribution can be defined. If ***F*** is a fuzzy subset of a universe of discourse ***U*** whose membership function is m_F than the value originated by the membership function could vary according to several degrees (or grade) of the membership m_F (u). ***R*** set can be defined ***associated with***[12] X (or fuzzy restriction on X) if it can be defined as an elastic constraint on the values that may be assigned to X. In this way a preposition as " the price could occur" can be expressed in a different way taking into account that X is the name of the object, a variable or a preposition and ***A*****(X)** is a particular attribute (comparability) of **X** which takes on different values in **U.** Then it will be possible to write:

$$\mathbf{R (A(X)) = F}$$

For example it considers the comparability of a property price assuming an ordinal scale U= [0,1,2,3,4,5] in this way, it is possible to write the relationship:

$$m_{comparable}(u) = 1 - S\,(u\ ;\ 0,\ 20{,}30)$$

Where **u** is the ordinal value of comparability, S is a function that links an ordinal value to the degrees of membership similarly to the set indicated in (d). They allow the appraiser to define a membership function which gives a number between 0 and 1 of comparability to different points in the ordinal scale.

In short it is possible to rewrite (e) as:

$$\mathbf{R(X) = F}$$

Then it will be possible to write that there is a possibility distribution $\mathbf{P}_X$ which is postulated to be equal to R(X). As a consequence there will be a possibility distribution function associated to it that can give a grade of membership for each value of **u**. This possibility distribution function can be considered as the membership function indicated in the (c) and (d) fuzzy sets.

Membership function could have different shapes which will give different results.

The possibility of the occurrence of a price is a measure question.

In order to show this difference, a case in which the appraiser is requested to appraise a property whose range of values could vary will be considered. As an example, the exihibit 1 will show different values that can be assumed by the property.

Numbers	**0**	**1**	**2**	**3**	**4**	**5**
Probable/Possible Prices	98,000	100,000	120,000	121,000	125,000	130,000

Exhibit 1- Different Possible or Probable Prices for a Real Property

It will be possible to define two different distributions: a probability distribution and a possibility one.

Distribution (*u*)	**0**	**1**	**2**	**3**	**4**	**5**
$\pi_{X\,(u)}$	0	0.2	0.5	0.7	0.8	1
$P_{X\,(u)}$	0	0	0.1	0.2	0.6	0.1

Exhibit 2 - A comparison between a possible distribution and a "subjective" probability distribution

Exhibit 2 shows the differences between the two different distributions. The “most probable” price will be the single point defined as the mode of a normal distribution of “subjective probability” at a certain level of probability. In this way the value of a property will be the number 3 with a 60% of probability. According to the Aristotelian point of view this evaluation is based on the dichotomy membership of an element to its set. In a different way the possibility distribution will offer for each price a “grade of membership”. This grades show how the price of the appraising property is "contained" inside the single element. In this case the most possible selling price is the number 5. In formal terms the probability distribution is based on the well known three following Kolmogorov's[13] axioms:

1. If the set A= 0 then P(A) = 0

2.If A is a subset of B then $P(A) \leq P(B)$

3. Probability is a **cardinal measure** of uncertainty and P(AunionB) = P(A) + P(B)

The possibility distribution relies on different basis. In particular, the first and the second axioms are the same both in the probability and in the possibility theory while the third axiom will change as follows:

3.Possibility is an ordinal measure of uncertainty and W(AunionB) = max(W(A),W(B)).

As one can observe the most possible and the most probable could not be the same and the information given by a possibility distribution is different than a probabilistic one. In the former distribution each element of the set has a different grade of comparability or membership to the set of a comparable property. In this case the difference seems to be small because both the grade of possibility and the percentage of probability are a subjective datum. But the appraiser is requested to define two different things. In the possibility distribution the appraiser shows how each comparable property can be defined similar to the one to be estimated, while in the probability distribution he tries to "put" the data inside a normal probabilistic distribution (according to the subjective probability). Furthermore they can be used in different contexts. In some cases a **cardinal** measure of uncertainty is possible. For example, an "event" can be estimated two or three times more probable than another using a **cardinal** measure of uncertainty. The appraiser uses this particular kind of approach in order to define a return measure or a subjective probability distribution. However there is also another nature of uncertainty in which a specific cardinal measure of occurrence of an event is not possible. Nobody knows if the probability of occurrence of an event is two or three times more than another. In this case the "possibility theory" offers an **ordinal** measure of uncertain events. Inside the possibility distribution is not clear if an event can be more or less probable than another.

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2. Property Appraisal and the "Most Possible Selling Price"

The relationship between the analyzed prices and the "possible value" can be described in several ways.

The graphic 1 below will consider the "grade of membership" of each value to the one to be estimated. Then the different grades of memberships will allow the appraiser to define how the appraising property is "contained" in each element of the set.

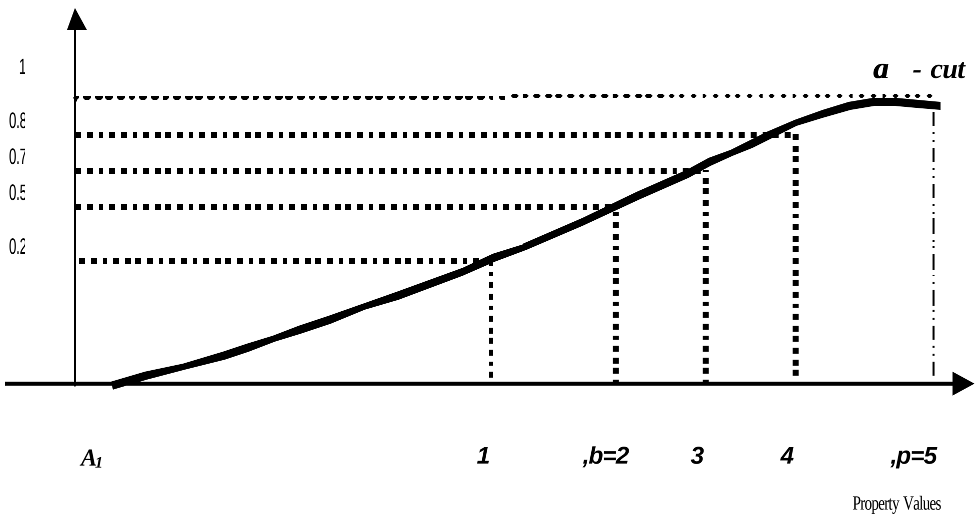

Figure 1 - The shape of a piece quadratic membership function

A possibility distribution can be described in several shapes: trapezoidal, triangular, piece quadratic. In this case, a piece quadratic membership function will be observed. Then the value of a property can be pointed out along the defined piece quadratic curve and identified by the appraiser starting from the market data, his personal experience or the other methodologies indicated in the previous paragraph.

The value given by the membership function will be the following formulas:

$$\mu_A = \begin{cases} \dfrac{1}{2\,(p - b - A_1)^2}\,(x - A_1) & \text{for} \quad 0 \le x \le 2 \\ -\dfrac{1}{2\,(b)^2}\,(x - p)^2 & \text{for} \quad 2 \le x \le 5 \end{cases}$$

The elements (price) of the set which have grade of membership equal to 1 are identical to that to be estimated. This function is the relationship between the observed measure and the different grade of membership "*that is characterized by a possibility distribution*"[14]. The shape of this distribution will vary depending on the shape which describes better the relationship between the elements of the set and their degrees of membership. The result will be defined on a level of confidence[15] that is indicated in the figure 1 as α-cut. As the appraiser is requested to define the "most possible value" with a more than 0.90 degree of possibility, then only the 5th value will be considered. If the appraiser is requested to define the most possible value with a degree of possibility more than 0.70. In this case the α - cut will define a different set whose members are:

$$A = \{3,4,5\}$$

In this interval (more than 0.70 of -cut) the price will vary between 121,000 and 130,000. In the same interval the "most possible selling price" (the value with the highest degree of possibility) is 130.000.

This means that an interval value can be estimated with a possibility distribution, too.

This concept is quite different from the normal probabilistic distribution on which is based the most probable selling price.

The fundamental concept of the "Most Probable Selling Price" has been introduced in the real estate literature by the Italian appraiser and professor Giuseppe Medici in his important work later translated into English[16] and strongly developed and improved by the American appraiser R. Ratcliffe[17].

The value of a real property can be defined as the "*most probable price which a property should bring in a competitive and open market under all conditions requisite to a fair sale, the buyer and the seller each acting prudently and knowledgeably, and assuming the price is not affected by undue stimulus*"[18] .

It has been stressed that in a "*statistical sense is a reference to the single most likely outcome (the mode), as opposed to the "expected price", which references the mean or average*"[19]. This is an important way to represent the value investigated by an appraiser. This concept is sometimes not consistent with Highest and Best Use of a property. In fact, it may occur that the most probable value is not the highest and best use of the real property. Simonotti[20] pointed the most probable selling price as the most important criterion for an appraisers while the highest and best use has a greater importance for a counselor.

In some case the appraiser is requested to define an interval of "probable" values for a property. In this context he has two different opportunities

In some appraisal reports in which the result is more uncertain the appraiser is used to defining a "range of value" and a probability distribution whose final result is the "most probable selling price".

From a probability perspective two different analysis are possible. If the frequentistic concept of probability will be chosen than the appraiser will collect several prices of properties comparable to the one to be estimated. The appraisal analysis will be successfully carried out analyzing the distribution of the prices. In the case of normal distribution, the most probable selling price will be the mean. Unfortunately, a normal distribution of prices is quite difficult to find, for this reason appraisers generally apply a subjective concept of probability defining a probability distribution which points for each price the relative probability of occurrence. In this way, the appraiser will define a normal probability distribution whose mean will be the "most probable selling price".

The highlighted differences between the "most probable" and the "most possible" lead to two different and useful remarks. The first concerns the general concept of the "most probable selling price". In this context the work will not try to demonstrate that the recurring definition of the most probable selling price is wrong. Probably there will be more than one perspective. In an important work Smith and Bagnoli pointed out that " *the final results of the fuzzy analysis will not produce Ratcliff's "probable price", since the mathematical process differs. The result should be similar, but more information is provided in terms of several "possible" prices having different degrees of membership…*"[21]

According to them and starting from the considerations defined in the previous paragraph a definition of the "most possible selling price" is proposed.

It can be viewed as "**the price that has the highest degree of a membership function inside the set of comparable properties when a cardinal measure of uncertainty is not possible**".

The second concerns an applicative case of appraisal. In the exhibit 3 there are two different possible destinations for an appraising property. The following properties have a destination A {0,1,2,3} while the property {4,5}. have destination B There are two different probability and possibility distributions.

	Destination A				Destination B	
Distribution (*u*)	**0**	**1**	**2**	**3**	**4**	**5**
$p_{x(u)}$	**0**	**0.2**	**0.5**	**0.7**	**0.8**	**0**
$P_{x(u)}$	**0**	**0**	**0**	**0.4**	**0.6**	**0**

Exhibit 3 - Probability and Possibility distribution of Prices

In this case the probability theory, according to third axiom of Kolmogorov, allows the appraiser to define the following relationships:

$$3 \cap 4 = 0 \Rightarrow P(3) \cup P(4) = P(3) + P(4) \Rightarrow E(\text{VALUE}) = P_3\ \text{Value}_3 + P_4\ \text{Value}_4$$

The final result of the appraisal report will be defined by the weighted sum between the different levels of probabilities and different values.

In this case, the application of the possibility theory will lead to different results, starting from the previous defined possibility distribution it will be possible to observe:

$$\Pi_x = \{(0,0)\ (0.2,1)\ (0.5,2)\ (0.7,3)\ (0.8,4)\} =$$
$$\max \pi(v)\ \{0\ ,0.2\ ,0.5\ ,0.7\ ,0.8\} = (0.8,4)$$

This final result will be defined through the greatest degree of possibility.

A cardinal measure of uncertainty can be defined better for those real estate market where the real property data are available and the appraiser can define a subjective or frequentistic distribution of probability. In some real estate market or for some specific properties the market data are not available, an ordinal measure of uncertainty seems to be more appropriate. In this context , the “most possible value” will be better.

Final Remarks and Future Directions of Research

At the end of this work the following aspects can be highlighted:

- Appraisal results are traditionally linked to the probabilistic concept of the “most probable selling price”, although there are some different growing approaches to uncertainty.
- The contribution and the difference between the most probable selling price and the most possible selling price has been defined and make evident.
- Using the most possible selling prices in appraisal reports seems to be more appropriate in those market with a lack of data and information. In fact the appraiser looks for the “most possible value” or a range of “possible values” starting from the grade of membership of each property price in comparison with the subject using an ordinal measure of uncertainty. Furthermore, possibility theory can not be based on fuzzy sets.
- The “most possible selling price” could give a further definition to return variability inside the Modern Portfolio Theory. It is possible to define a variability in terms of “Possibility Theory”.

This seems important in real estate markets where it is necessary to start from the subjective point of view[22]

A further direction for the research could be to explore the contribution of the **Imprecise Probability** in the real estate appraisal and the Modern Portfolio Theory applied to real property assets

References

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