GEOMETRY OPTIMIZATION OF TENSEGRITY STRUCTURES WITH SPREADSHEET SOFTWARE

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ABSTRACT

Many branches are inspired by tensegrity as a structural principle since Richard Buckminster Fuller created this neologism from tens(ion) and (int)egrity 60 years ago.

The apparently floating bars in a cable net suggest a great lightness. This is a very interesting feature for architects and engineers.

However, no widespread application of tensegrity structures can be found in architecture as yet. Reasons for this are the complex geometrical and mechanical properties.

It is nearly impractical to give fast information about the general feasibility of architectural concepts.

This paper presents firstly a tool employing the spreadsheet software Microsoft Excel which can give such information. Therefore it is not necessary to purchase special software and the according time consuming training is much lower.

The geometry optimization of two spatial tensegrity basis modules, using the Excel Solver, is described in the second part of the paper.

KEY WORDS

Tensegrity, geometry checking, geometry optimization, Excel Solver, pre-stress

TOOL TO CHECK GEOMETRY AND TOPOLOGY

The geometry and topology design concept of space frames comprises fundamental information about the mechanical properties. Especially for the design of tensegrity structures, as a subclass of space frames, it is essential to control if the structure is stable and if pre-stress can be applied.

These questions can be answered based on the extension of the well known Maxwell's rule.

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MAXWELL'S RULE

The number of bars, which are at least necessary to stabilize a space frame with frictionless joints, can be determined by Maxwell's rule.

The rule for the construction of rigid two-dimensional frameworks with b bars and j frictionless joints is:

$$b = 2 * j - 3$$

and:

$$b = 3 * j - 6$$

for rigid three-dimensional frameworks.

Maxwell himself assumed exceptions of his rule. He anticipated that stiff structures maybe possible with a smaller number of bars and also cases, where the structure is free to move even if his rule is satisfied (Maxwell 1864).

THE EXTENDED MAXWELL'S RULE

In the seventies of the 20th century Calladine (Calladine 1978) went back to the exceptions of Maxwell's rule. The result from his study is the extended version of Maxwell's rule, which includes all possible cases:

$$3 * j - b - c = m - s$$

Therein c is the number of cinematic constraints ($c \geqq 6$ in three dimensions), m the number of internal mechanisms and s the number of self-stress states.

The two dimensional example in Figure 1 illustrates the advancement of the rule.

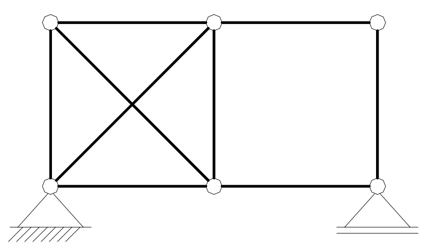

Figure 1 Two dimensional structure which satisfies Maxwell's rule but includes a mechanism

The structure has six joints, three cinematic constraints and nine bars, so Maxwell's rule is exactly fulfilled. Despite this fact the structure obviously is not stable and includes one mechanism m and one self-stress state s. The self-stress state can not stabilize the whole structure. Configurations like this can be checked with the presented tool as described in the following.

DETERMINATION OF m AND s

The key value for the calculation of m and s is the rank of the equilibrium matrix of the nodes. The largest number of linear independent column vectors and the largest number of linear independent row vectors are always equal. This number r_A is called rank of the matrix. The equilibrium matrix A is a (m x n) matrix, where m is the number of joints (j * 3) and n is the number of unknown element forces. A contains the direction cosines, in the x, y, z directions, of each element. One feasible method to determine the rank of a matrix is the singular value decomposition (SVD). For this the number of non-zero singular values of a matrix is identical with the rank of this matrix. Detailed information on this operation and on its application to the equilibrium matrix is given in (Pellegrino 1993).

After the rank determination, m and s can be calculated with the following formulas:

$$s = b - r_A$$
$$m = 3 * j - c - r_A$$

The values of m and s depend not only on the number of bars and joints, nor even on the topology of the structure, but essential on the complete geometry (Pellegrino 1986).

IMPLEMENTING IN EXCEL

The Excel add-in was developed using visual basic for applications (VBA) as programming language. The add-in can be employed to calculate the above described values m and s for general plane and spatial structures. An Excel worksheet works as user interface. The required input data are the node coordinates, the cinematic constraints and the topology of the connections. Using a virtual reality modelling language (VRML) plug in, it is possible to visualize the input data of the structure in any internet browser.

The VBA algorithm formats the node equilibrium matrix based on the input data. Subsequently the SVD of this matrix is carried out and the number of non-zero singular values can be achieved. One has to note that none of this values is actually equal to zero, but some are much smaller than others. Pellegrino (Pellegrino 1993) recommends to treat values as zero if they are smaller than 10^{-3} after being multiplied with the largest singular value. This is valid for the most structural assemblies and therefore basis for the rank determination in the VBA algorithm.

The output of the values m and s will be populated in an worksheet.

RELEVANCE OF m AND s

A summary of the information about the fundamental mechanical properties of space frames, that can be derived from the values m and s, is given in Table 1.

Table 1 Information about the fundamental mechanical properties on basis of m and s

1	m = 0 s = 0	The structure is stable and can not be pre-stressed. → useable
2	m > 0 s = 0	The structure is not stable and can not be pre-stressed. → unuseable
3	m = 0 s > 0	The structure is stable and can be pre-stressed. → useable
4	m > 0 s > 0	The structure has mechanisms and can be pre-stressed. → conditionally useable (further investigation necessary)

Tensegrity structures are mainly assigned to the fourth row of Table 1. They are considered to be cinematically indeterminate ($m > 0$) and statically indeterminate ($s > 0$). The vector of the element pre-stress forces (t_0) has to be checked in order to clarify if the geometry is in a stable equilibrium or not. This vector t_0 is filled with the element forces for the loading case pre-stress and must be a solution for the following homogeneous equation system.

$$A * t_0 = 0$$

It is to be examined whether element forces in t_0 are equal to or approximately zero. If this is not the case, the available mechanisms can be stabilised by pre-stress.

While using cables, because of their unilateral rigidity, it must be additionally checked if the cables are tensioned and the bars are subjected in compression.

As mentioned above, the SVD of the equilibrium matrix in the described Excel add-in is used for the rank determination. This decomposes the matrix A to:

$$A = U * W * V^T$$

For a (m x n) matrix the m left singular vectors are returned in U and the n right singular vectors are returned in V. The matrix W is diagonal and contains the singular values of A.

The right singular vectors are of special interest. All right singular vectors with a belonging singular value equal to zero are self-stress states (Pellegrino 1986). The self-stress state is identical with t_0 only for the case $s = 1$. The self-stress state is then plotted to a worksheet.

EXAMPLE: DESIGN OF A FREE – FORM TENSEGRITY STRUCTURE

As an example of application of the tool the design of a free – form tensegrity structure is presented. The purpose of the building is an aviary for the Zoological Garden of Rostock, Germany. It was designed by the architect Klaus Wagener and the structural planning is from Veit Bayer. Figure 2 shows a model of the project which is at length described in (Bayer 2004).

A topology check shows the aviary has 16 joints and 18 cinematic constraints. In order to be stable it must have (3 * 16 – 18 = 30) elements according with Maxwell's rule. The design concept plans, however, only 28 elements. A geometry check has to clarify if the structure can be pre-stressed and if this pre-stress can stabilize the whole structure.

The start geometry was changed step by step by variation of the node coordinates until a geometry was found which can be pre-stressed. The final geometry has two mechanisms (m = 2) and one self-stress state (s = 1) which stabilizes the two mechanisms.

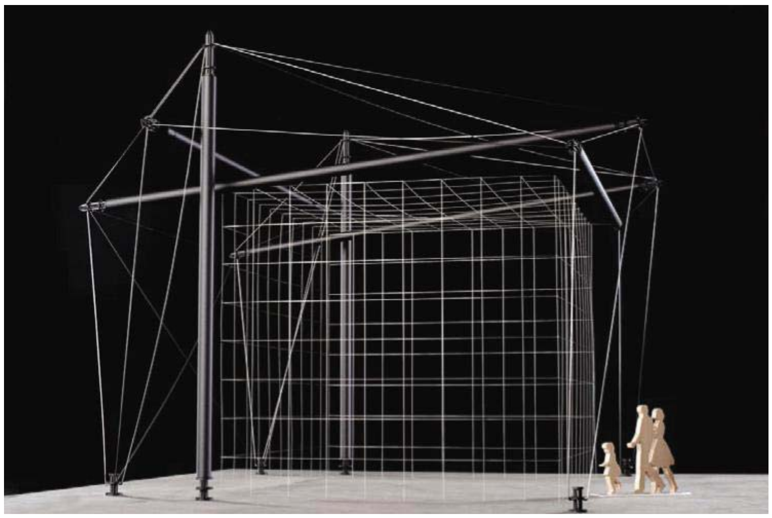

Figure 2 Model of the example, a tensegrity aviary. Photo courtesy of K. Wagener, Berlin and V. Bayer, Weimar.

GEOMETRY OPTIMIZATION

The Geometry of tensegrity structures has fundamental importance to their mechanical behavior. It was shown in the first part of this paper that only certain geometries have a stable and pre-stressable equilibrium. This equilibrium condition must always be fulfilled during optimization of the geometry.

The presented geometry optimization has been carried out on two modules, which have similar shape but differ a lot in load bearing behavior.

Topology I is the most simple spatial tensegrity basis module. It consists of three bars and nine cables. The so called three-bar module is characterized by a twist-angle (α) of 30° between upper and lower base polygons. This module comprises one mechanism ($m = 1$) and one state of self-stress ($s = 1$).

The independent variables of the optimization are the vertical distance between the base polygons (H) and the ratio of the length of the upper and the lower polygon cables ($l_{low} / l_{up} = R$). For $R < 1$ the module grows broader and for $R > 1$ the module is tapering towards the top. Stable geometries can only be achieved with $\alpha = 30°$.

Topology II is like topology I, but owns three additional cables. It comprises no mechanism and three states of self-stress. The independent variables of the optimization are H and R again and the twist-angle between upper and lower base polygons (α). In contrast to variant I stable geometries are possible with $\alpha \neq 30°$. This is permitted by the three additional cables.

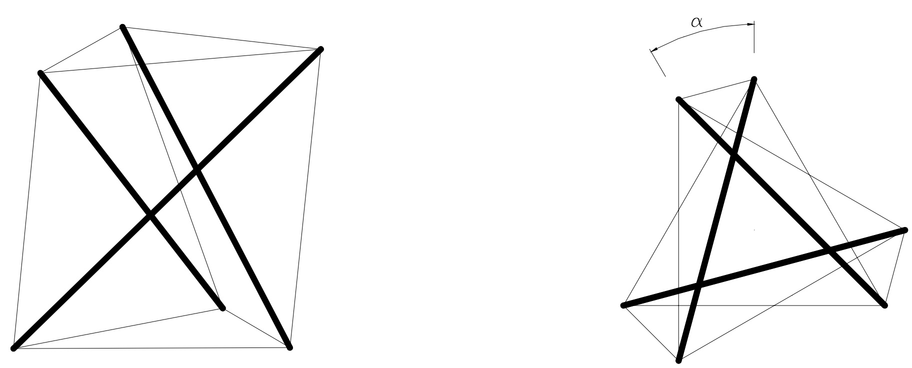

Figure 3 Isometry and top view of the three-bar module.

REMARKS TO THE EXCEL SOLVER

The standard Solver which is included in the MS Office Package is able to solve linear and non-linear equation systems. The simplex method is employed for linear problems and a special gradient method for non-linear problems respectively.

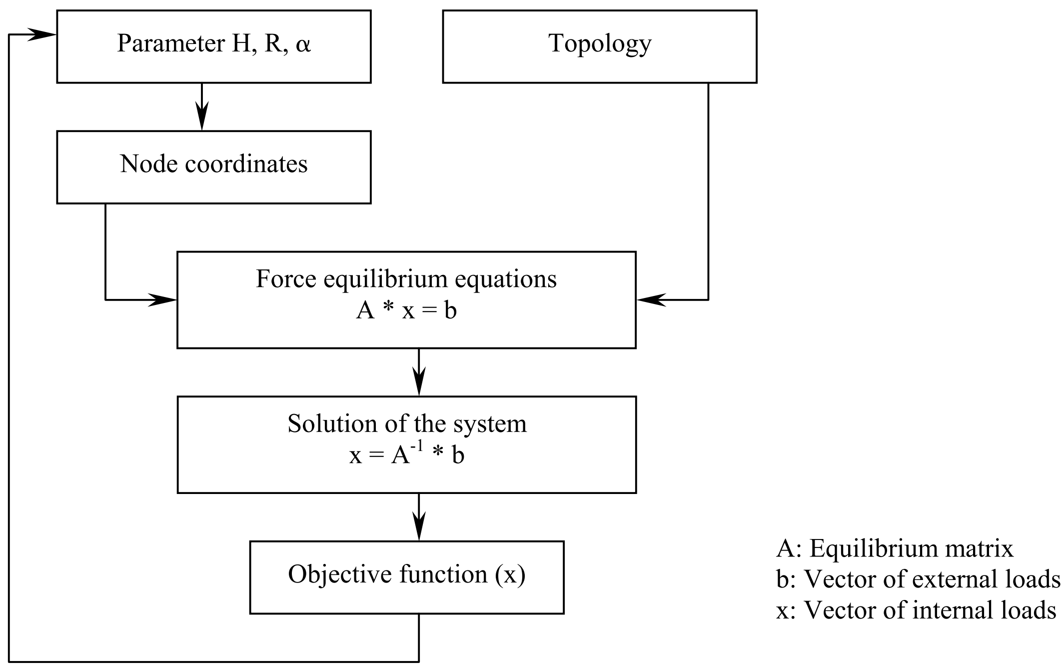

Figure 4 Workflow of the optimization in a worksheet.

The Excel Solver requires all constraints, variables and the objective to be in an Excel worksheet. Variables or arrays defined by a VBA macro can not be processed by the Solver. (Steidel 2006)

The optimization parameters and the objective function therefore have to be linked by cell references. The workflow of the optimization in a worksheet is shown in Figure 4.

The optimization uses the internal element forces given by the solution of the linear equation system:

$$A * x = b$$

If the solution of this system would be done by means of a matrix function programmed in VBA, the cells would just be filled with the values of A and x without any remaining cell reference. That's why it is necessary to fill cell by cell with the formulas of the matrix function. This substantially limits the size of the equation system which can be processed in justifiable time. For the here discussed modules the symmetry of the structure was utilized to reduce the matrix size.

FORMATION AND SOLUTION OF THE EQUATION SYSTEM

For any structure the equilibrium matrix of the nodes A is rectangular with the size (m x n). The number of nodes gives m (m = 3 * j) and the number of elements gives n (n = b). For the chosen variants of topology A has a dimension of (18 x 12) and (18 x 15) respectively.

Topology I:

The three nodes in each base polygon are the same. One bar and three cables, thus four elements, are connected on each node. The element force of one element, connected to the node, is set to one. Hence are three equations and three unknown element forces existing on each node. However, two of the three equations are linear dependent. That leads to a determinant of the equilibrium matrix equal to zero. Actual only two equations are available. A third equation is provided by the symmetry of the base polygons. For the loading case pre-stress the element force is equal in all cables of one base polygon. The equilibrium matrix of the nodes is consequently regular and quadratic and the equation system (A * x = b) can be solved with the inverse equilibrium matrix (A^{-1}).

$$x = A^{-1} * b$$

The determination of all unknown element forces requires the solution of the node equilibrium of one upper and one lower node.

Topology II:

Here also two sets of three equal nodes in the base polygons are existing. However, one bar and four cables, thus five elements, are connected on each node. Because of the twist-angle $\alpha \neq 30°$ is the determinant of the equilibrium matrix unequal to zero and are all equations linear independent. Four equations exist for four unknown element forces. The further procedure is like in topology I.

OBJECTIVE FUNCTIONS

The determined element forces are the arguments of the objective function. The value of the element forces are dependent on the described optimization variables.

Homogeneous distribution of pre-stress as objective

Pre-stress is one of the proper loading case for tensegrity structures (Schlaich 2003). Similar to membrane structures is a homogeneous material strain over the whole structure advised for this loading case. The homogeneous distribution of pre-stress was mathematical formulated as a minimization of the standard deviation of the element forces in the cables.

$$\min f(x) = \sqrt{\frac{n\sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i\right)^2}{n^2}}$$

Herein n is the number of cables and x_i the value of the element force.

Because of the principle workflow of the optimiziation in an worksheet it is easy to formulate different other objective functions. For the load case pre-stress for example the minimization of the total volume of the structure with and without consideration of buckling risk of the bars was examinated.

The exactly description of these objective functions, the influence of constraints and the analysis of the results of the optimization are out of the scope of this article. These topics are at length pursued in (Wolkowicz 200x).

CONCLUSIONS

The question of stability is certainly the first that crops up when tensegrity structures are to be designed.

A software tool was therefore developed, that gives information about the stability while using only standard software and freeware.

The geometry influences the properties of tensegrity structures in a crucial manner. On example of two basis modules a geometry optimization in a worksheet of a standard spreadsheet software was described.

The presented tool is able to support the structural designer in an early stage of the project in finding a feasible architectural concept for tensegrity structures.

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